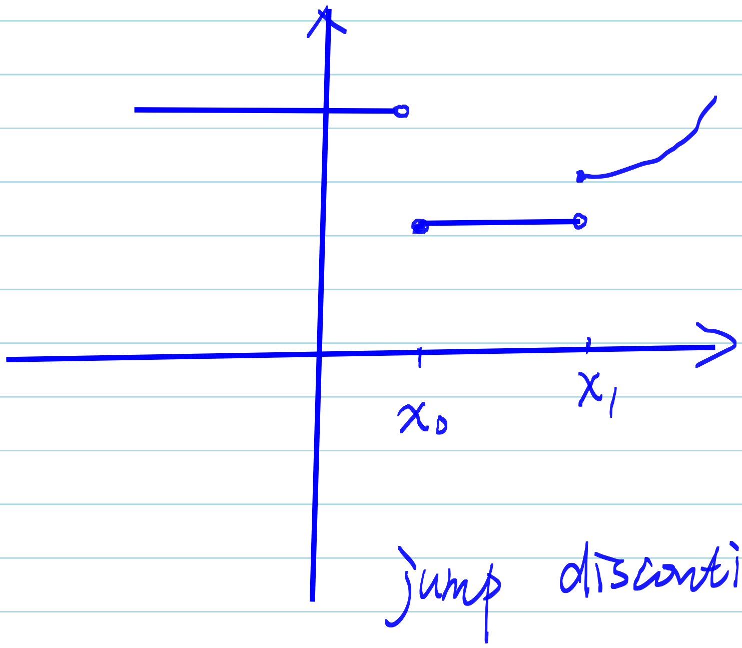
Lecture 17

Plan: §7.6 Laplace transform of discontinuous function

Recall: $\begin{cases} f \text{ piecewise continuous on } [0, \infty) \\ f: \text{exponential order } \alpha. \end{cases}$

$\Rightarrow \mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Remark: f needs NOT be continuous everywhere for us to study $\mathcal{L}\{f\}$.



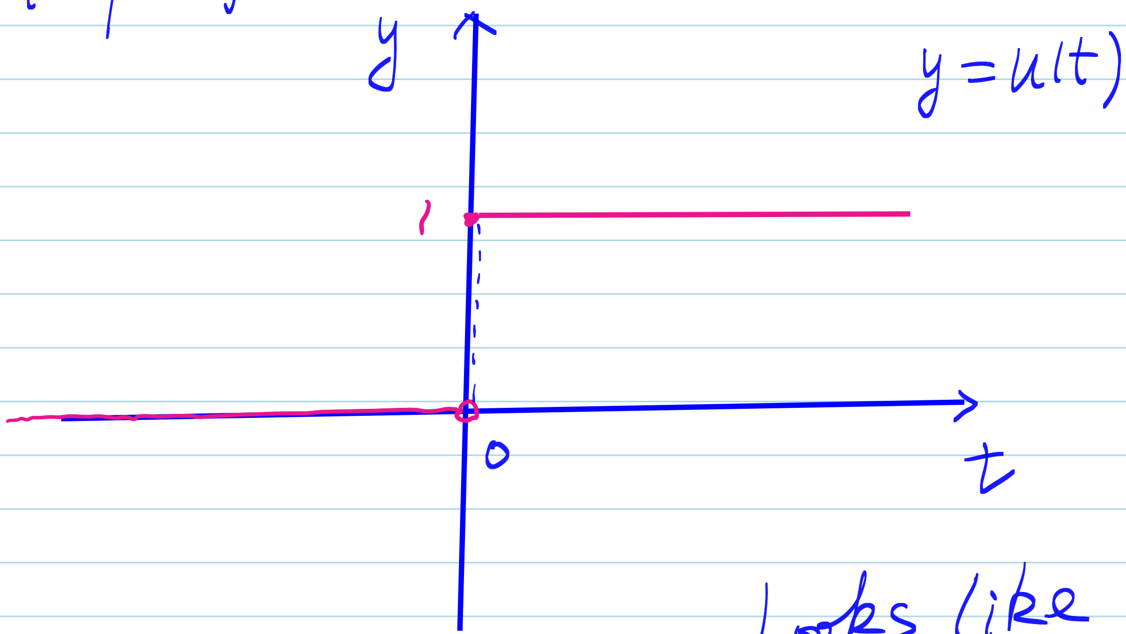
jump discontinuous at x_0, x_1

An important discontinuous (but piecewise continuous) function is the unit step function $u(t)$.

Defn: The unit step function $u(t)$ is defined by

$$u(t) = \underbrace{\begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}}_{\text{piecewise defined}}$$

Graph of $u(t)$:



Looks like a step/staircase

$$\begin{cases} u(x) \\ x = t-a \end{cases}$$

Q: Given a constant $a \in \mathbb{R}$, what does the function $u(t-a)$ look like?

A: Recall

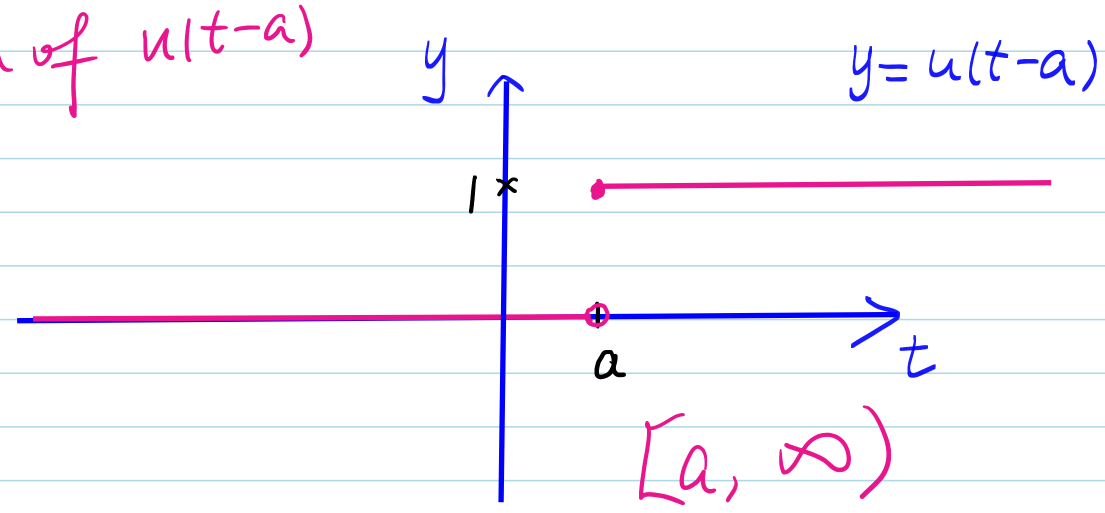
$$u(x) = \begin{cases} 0 & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Replace all "x" by "t-a". $\Rightarrow$

$$u(t-a) = \begin{cases} 0, & t-a < 0 \\ 1, & t-a \geq 0 \end{cases}$$

$$= \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Graph of $u(t-a)$



Def^n: Given two constants $b > a$.

The window function $\Pi_{a,b}(t)$ is defined by

$$\Pi_{a,b}(t) = u(t-a) - u(t-b).$$

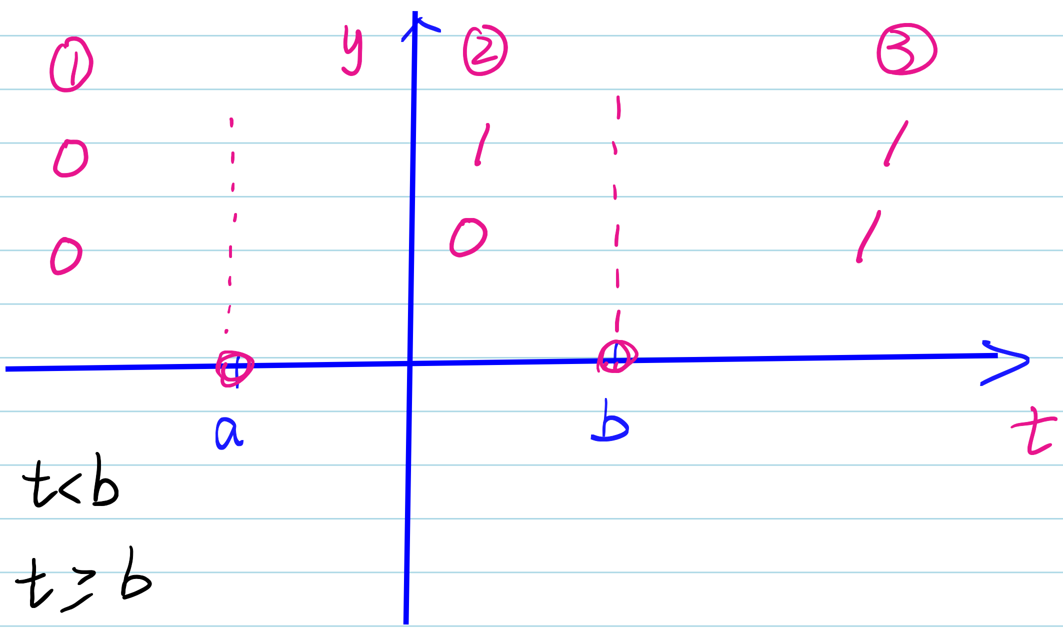
Capital π

Remark:

$$\Pi_{a,b}(t) = \begin{cases} 0 & t < a \\ 1 & t \in [a,b) \\ 0 & t \geq b \end{cases}$$

Why? Recall $\Pi_{a,b}(t) = u(t-a) - u(t-b)$.

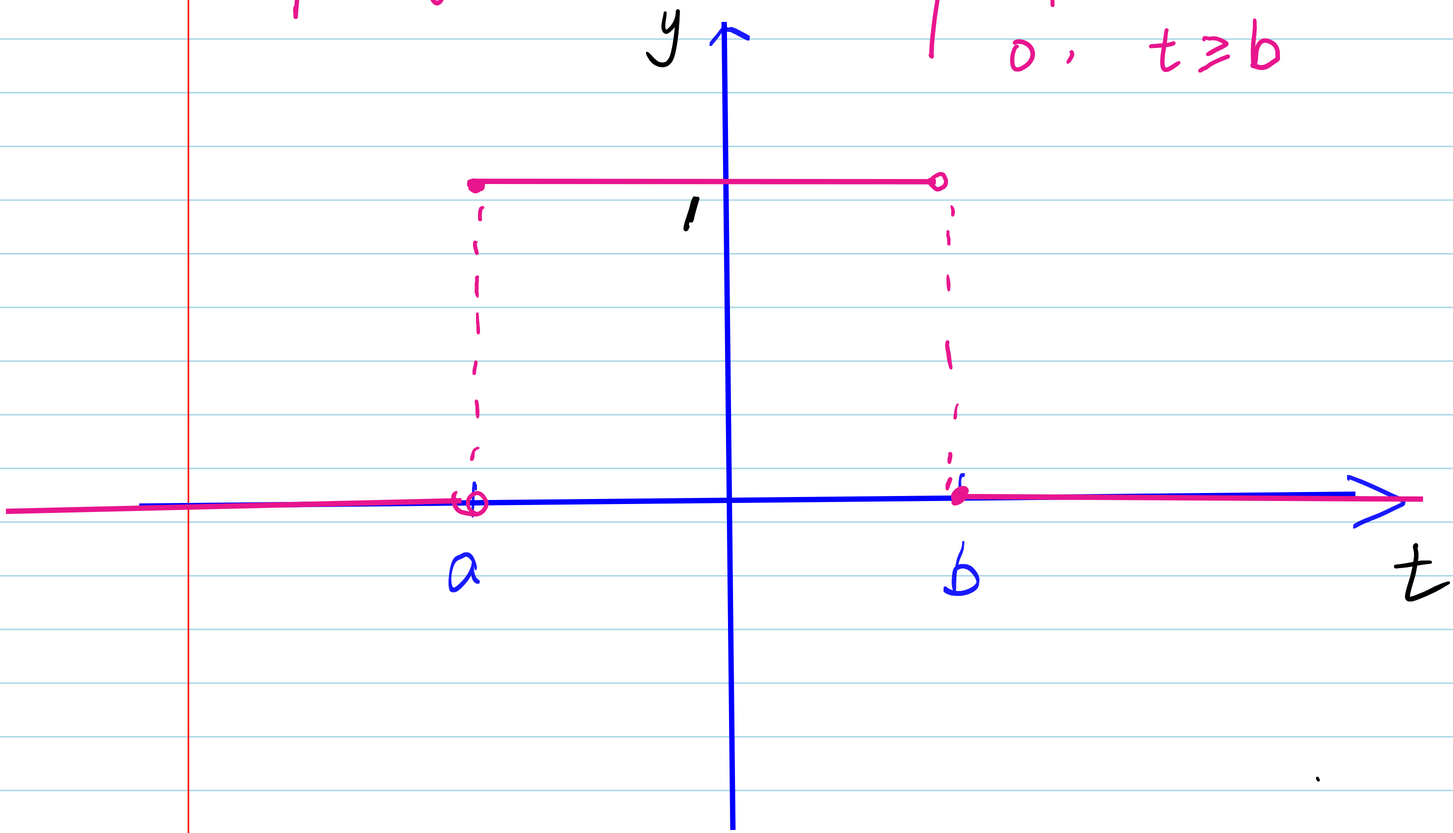
	①	②	③
$u(t-a)$:	0	1	1
$u(t-b)$:	0	0	1



$$u(t-b) = \begin{cases} 0, & t < b \\ 1, & t \geq b \end{cases}$$

Q: Why is $\Pi_{a,b}(t)$ called the "window" function?

A: Graph of $\Pi_{a,b}(t)$: $= \begin{cases} 0, & t<a \\ 1, & t\in[a,b) \\ 0, & t\geq b \end{cases}$



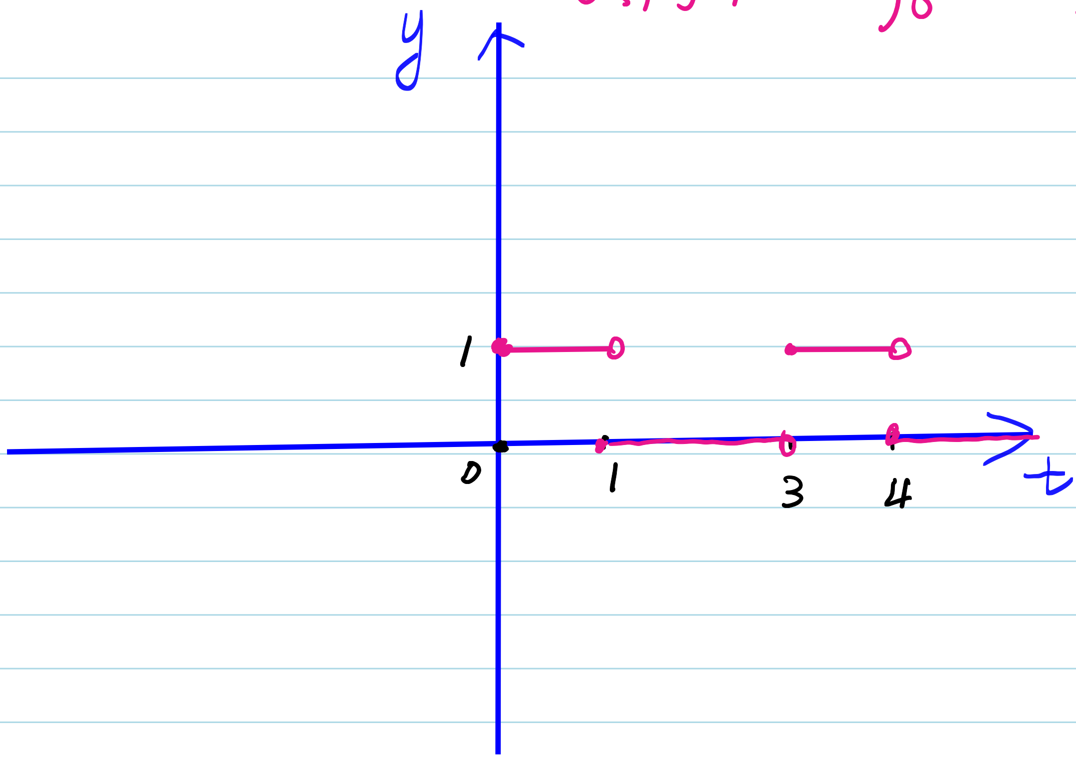
E.g. ① plot $y(t) = \Pi_{0,1}(t) + \Pi_{3,4}(t)$ for $t \geq 0$

② plot $y(t) = 2\Pi_{0,1}(t) + 3\Pi_{3,4}(t)$ for $t \geq 0$

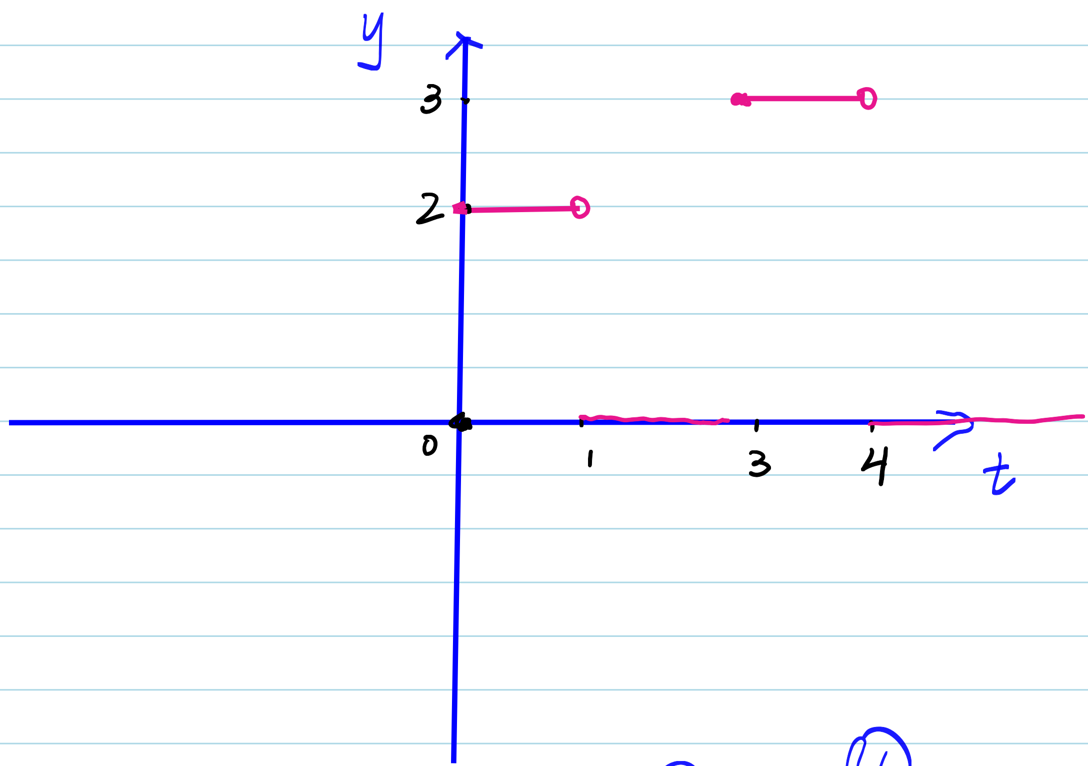
Hint: $\Pi_{a,b}(t)$ contributes 1 over the interval $[a,b)$, and contribute nothing (0) at other places.

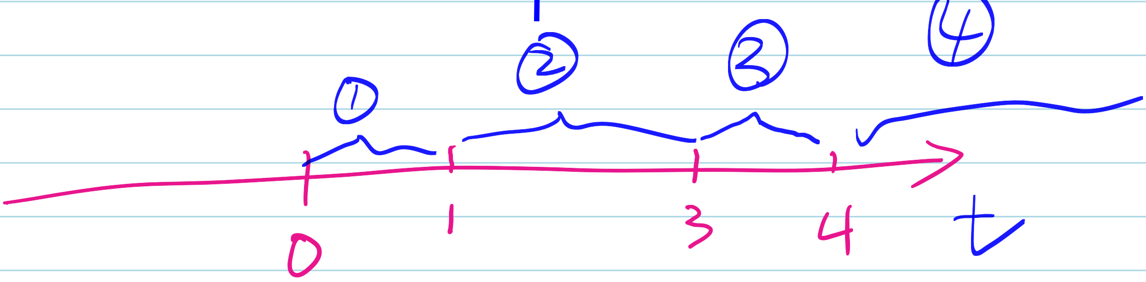
$$\mathcal{L}\{f\}(s)=\int_0^\infty e^{-st}f(t)\,dt$$

①



②

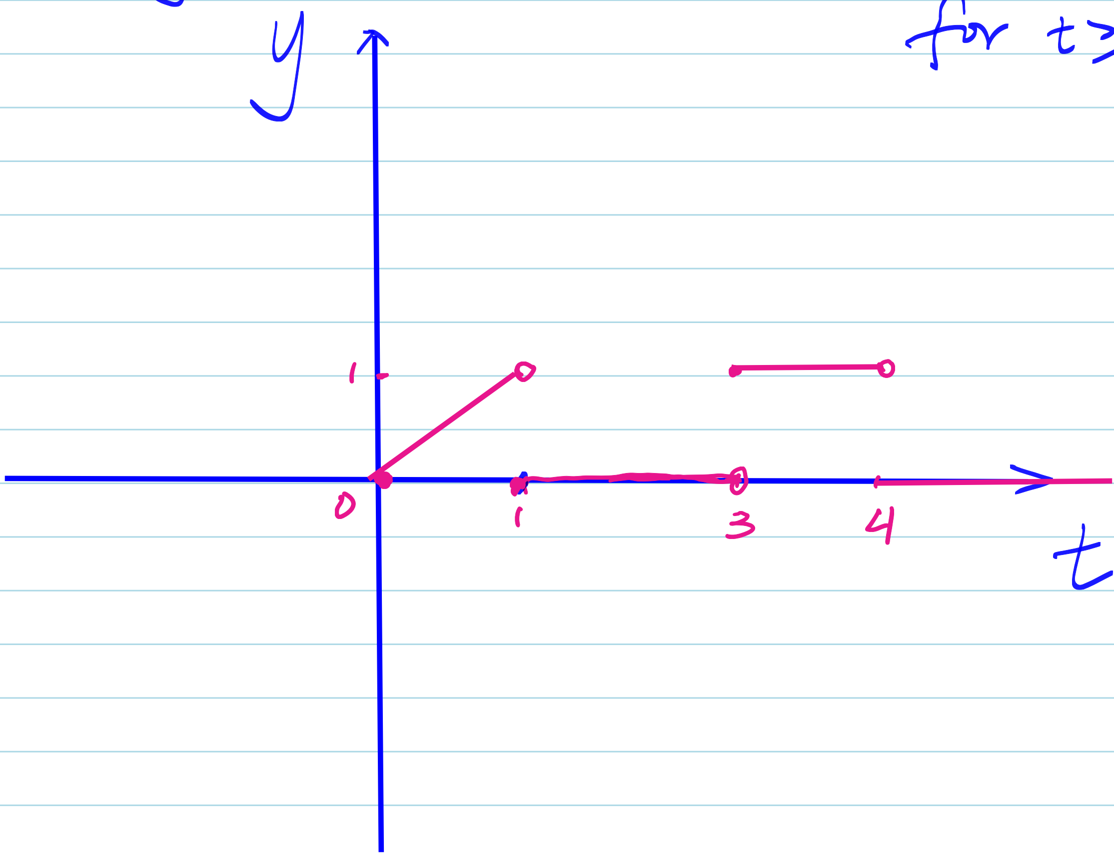




Plot the graph of

E.g.: $y(t) = t\,\Pi_{0,1}(t) + \Pi_{3,4}(t)$

for $t \geq 0$



Summarize:

① $g(t)\,\Pi_{a,b}(t)$ contributes $g(t)$ to $[a,b)$, and contributes 0 to other places.

② $h(t)\,u(t-a)$ contributes $h(t)$ to $[a,\infty)$, and contributes 0 to other places.

E.g Let

$$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 1, & t \in [2,5) \\ t, & t \in [5,8) \\ t^2/10, & t \ge 8 \end{cases}$$

write f in terms of the window function and the step function

A:

$$f(t) = 3\Pi_{0,2}(t) + \Pi_{2,5}(t) + t\,\Pi_{5,8}(t) + \frac{t^2}{10}u(t-8)$$

function \ interval	$[0,2)$	$[2,5)$	$[5,8)$	$[8,+\infty)$
$3\Pi_{0.2}(t)$	3	0	0	0
$\Pi_{2.5}(t)$	0	1	0	0
$t\Pi_{5.8}(t)$	0	0	t	0
$\frac{t^2}{10}u(t-8)$	0	0	0	$\frac{t^2}{10}$
$f(t)$	3	1	t	$t^2/10$

Proposition:

① Given $a \geq 0$, then

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s} \quad \text{for } s>0$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}(t) = u(t-a)$$

② For $b \geq a \geq 0$,

$$\mathcal{L}\{\Pi_{a,b}(t)\}(s) = \mathcal{L}\{u(t-a) - u(t-b)\}$$
$$= \frac{e^{-as} - e^{-bs}}{s}$$

③ Assume $F(s) = \mathcal{L}\{f\}(s)$ exists for $s > \alpha \geq 0$. If $c > 0$, then

$$\mathcal{L}\{f(t-c)\,u(t-c)\}(s) = e^{-cs}F(s), \quad s > \alpha$$

and $\mathcal{L}^{-1}\{e^{-cs}F(s)\}(t) = f(t-c)\,u(t-c)$.

Pf: ①

$$\mathcal{L}\{u(t-a)\}(s)$$

$$= \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} \underbrace{\cancel{u(t-a)}}_{0} dt + \int_a^\infty e^{-st} \underbrace{\cancel{u(t-a)}}_{1} dt$$

$$= \int_a^\infty e^{-st}\, dt$$

$$= \lim_{N\to\infty} \int_a^N e^{-st}\, dt$$

$$= \lim_{N\to\infty} \left.\frac{-e^{-st}}{s}\right|_a^N = \frac{e^{-as}}{s}$$

②

$$\mathcal{L}\{\Pi_{a,b}(t)\}(s) = \mathcal{L}\{u(t-a) - u(t-b)\}(s)$$

$$= \mathcal{L}\{u(t-a)\}(s) - \mathcal{L}\{u(t-b)\}(s)$$

$$= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

$$= \frac{e^{-as} - e^{-bs}}{s}$$

Pf of ③: Read the book.

E.g 1: compute $\mathcal{L}\{(\cos t)(u(t-\pi))\}$

A:

Idea: find $f(t)$ such that

$$f(t-\pi) = \cos t. \qquad (*)$$

How to find such $f(x)$?

let $x = t-\pi$ in $(*)$. $\Rightarrow t = x+\pi$

Hence $(*)$ becomes

$$f(x) = \cos(x+\pi)$$

or $f(t) = \cos(t+\pi)$

Then

$$\mathcal{L}\{\cos t\, u(t-\pi)\}$$

$$= \mathcal{L}\{f(t-\pi)\, u(t-\pi)\}$$

$$= e^{-\pi s}\, \underbrace{\mathcal{L}\{f(t)\}}_{} \overset{\text{F}}{/\!/}$$

$$= e^{-\pi s}\, \mathcal{L}\{\cos(t+\pi)\}$$

$$= e^{-\pi s}\,\mathcal{L}\{-\cos t\}$$

$$= -e^{-\pi s}\,\mathcal{L}\{\cos t\}$$

$$= -e^{-\pi s}\,\frac{s}{s^2+1}$$

Hint:

$\cos(t+\pi) = -\cos t$

Table §7.2

$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

E.g 2: Compute $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$.

A: Idea: $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\underset{\downarrow\ c=2}{e^{-2s}} \cdot \underbrace{\frac{1}{s^2}}_{F}\right\}$

Let $F(s) = \frac{1}{s^2}$.

We first find $f(t) = \mathcal{L}^{-1}\{F\}(t)$:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t$$

↑ Table in §7.2

By proposition ③,

$$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{s^2}\right\} = \mathcal{L}^{-1}\{e^{-2s}\cdot F(s)\}$$

$$= f(t-2)\,u(t-2) = (t-2)\,u(t-2).$$

$f(t) = t \Rightarrow f(t-2) = t-2$

E.g 3: Compute $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\}$.

A: Note $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\underbrace{e^{-s}}_{c=1} \cdot \underbrace{\frac{1}{s(s^2+4)}}_{F(s)}\right\}$

Let $F(s) = \frac{1}{s(s^2+4)}$

Then we need to find $f(t) = \mathcal{L}^{-1}\{F\}(t)$

For that, we find the partial fractional decomposition of F:

$$F(s) = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$\text{-----} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = 0 \end{cases}$$

Ex. see Lecture 15

Hence $F(s) = \frac{1}{4}\frac{1}{s} - \frac{\frac{1}{4}s}{s^2+4}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s} - \frac{\frac{1}{4}s}{s^2+4}\right\}$$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

Table §7.2

$$= \frac{1}{4} - \frac{1}{4}\cos(2t)$$

Then by proposition ③,

$$\mathcal{L}^{-1}\{e^{-s}F(s)\} = f(t-1)\,u(t-1)$$

Q: What is $f(t-1)$?

A: Recall $f(t) = \frac{1}{4} - \frac{1}{4}\cos(2t)$

Replace all "t" by "$t-1$":

$$f(t-1) = \frac{1}{4} - \frac{1}{4}\cos 2(t-1).$$

Hence

$$\mathcal{L}^{-1}\{e^{-s}F(s)\} = \left(\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right)u(t-1).$$

E.g 4. Let $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 2\cos t & t \ge \pi \end{cases}$

Find $\mathcal{L}\{f(t)\}$.

First write f using step ~~and/or window functions~~

A: Note $f(t) = 2\cos t\, u(t-\pi)$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2\cos t\, u(t-\pi)\}$$

$$= 2\mathcal{L}\{\cos t\, u(t-\pi)\}$$

$$= -2e^{-\pi s}\frac{s}{s^2+1}.$$